The boundary conditions are then

$$\bar{u}[x,y,f(x,y)] = v[x,y,f(x,y)] = 0$$
$$\bar{u}(x,y,\infty) = 1 \qquad v(x,y,\infty) = 0$$

These become independent of $f(x, y)$, and the equations are simplified on introducing the transformation

$$d\theta = F(x,y)dx \qquad \phi = \psi - f(x,y)$$

Equations (7) and (8) become

$$\frac{\partial \bar{u}}{\partial \theta} + G(\theta,y)\frac{\partial \bar{u}}{\partial \phi} = \frac{1}{u\,\bar{u}}\left(\frac{\rho_e}{\rho} - \bar{u}^2\right)\frac{\partial u_e}{\partial \theta} + \frac{\partial}{\partial \phi}\left(\bar{u}\frac{\partial \bar{u}}{\partial \phi}\right) \quad (9)$$

$$\frac{1}{r}\frac{\partial}{\partial \theta}(vr) = \frac{Ku_e}{F\bar{u}}\left(\frac{\rho_e}{\rho} - \bar{u}^2\right) + \frac{\partial}{\partial \phi}\left(\bar{u}\frac{\partial v}{\partial \phi}\right) \quad (10)$$

in which $G(\theta, y) = r\rho_w w_w/\rho_0 F$, and the boundary conditions are $\bar{u}(\theta, y, 0) = v(\theta, y, 0) = v(\theta, y, \infty) = 0, \bar{u}(\theta, y, \infty) = 1$

As usual, on applying the solution (5) to the boundary layer in a perfect gas we have

$$(\rho_e/\rho) - \bar{u}^2 = [1 + (\gamma - 1)M_e^2/2](1 - \bar{u}^2) = c(a_0/a\,)^2(1 - \bar{u}^2)$$

where $c = 1 + (\gamma - 1)M_0^2/2$, M is the Mach number, and Eqs (9) and (10) become, finally,

$$\frac{\partial \bar{u}}{\partial \theta} + G(\theta,y)\frac{\partial \bar{u}}{\partial \phi} = c\left(\frac{a_0}{a_e}\right)^2\frac{(1 - \bar{u}^2)}{\bar{u}}\frac{\partial}{\partial \theta}(\log u_e) + \frac{\partial}{\partial \phi}\left(\bar{u}\frac{\partial \bar{u}}{\partial \phi}\right)$$

$$\frac{1}{r}\frac{\partial}{\partial \theta}(vr) = \frac{Ku_e c}{F}\left(\frac{a_0}{a_e}\right)^2\frac{(1 - \bar{u}^2)}{\bar{u}} + \frac{\partial}{\partial \phi}\left(\bar{u}\frac{\partial v}{\partial \phi}\right)$$

For the special case of incompressible flow denoted by suffix i with density ρ_0 and viscosity μ_0, these equations become

$$\frac{\partial \bar{u}_i}{\partial \theta} + G_i(\theta,y)\frac{\partial \bar{u}_i}{\partial \phi} = \frac{(1 - \bar{u}_i^2)}{\bar{u}_i}\frac{\partial}{\partial \theta}(\log u_{ei}) + \frac{\partial}{\partial \phi}\left(\bar{u}_i\frac{\partial \bar{u}_i}{\partial \phi}\right)$$

$$\frac{1}{r}\frac{\partial}{\partial \theta}(v_i r) = \frac{K_i u_{ei}(1 - \bar{u}_i^2)}{F_i \bar{u}_i} + \frac{\partial}{\partial \phi}\left(\bar{u}_i\frac{\partial v_i}{\partial \phi}\right)$$

with the same boundary conditions on $\bar{u}_i$, v_i as on $\bar{u}$, v

Hence, the two sets of equations for $\bar{u}$, v and $\bar{u}_i$, v_i have identical solutions as functions of (θ, y, ϕ), provided that

$$c\left(\frac{a_0}{a}\right)^2\frac{\partial}{\partial \theta}(\log u_e) = \frac{\partial}{\partial \theta}(\log u_i) \quad (11)$$

$$G(\theta,y) = G_i(\theta,y) \quad (12)$$

$$\frac{Ku_e c}{F}\left(\frac{a_0}{a_e}\right)^2 = \frac{K_i u_{ei}}{F_i} \quad (13)$$

The correlation is therefore established if the main streams of the compressible and incompressible flows are related by (11), a suitable integral of which is

$$u_i = a_0 u_e/a_e$$

the wall velocities are related by (12), yielding

$$w_{wi} = \mu_0 a_0 w_w/\mu_w a_e$$

and the values of K in the two flows are related by (13), giving

$$K_i = c\mu_0\rho_0 a_0^2 K/\mu_w\rho_w\, a_e^2$$

which is equivalent to Cooke's result

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Analogy between Three-Dimensionally Heated Plates and Generalized Plane Stress

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CONSIDER a thin plate of arbitrary contour C which is subject to in-plane body forces per unit of area $[X(x, y), Y(x, y)]$ where x and y are midplane coordinates Assuming that the displacements are prevented on the boundary, the principle of stationary potential energy for this system requires that[1]

$$\delta V^* = \delta \iint [V_0 - (Xu + Yv)]dxdy = 0 \quad (1)$$

where V^* is the potential energy, and V_0 is the strain energy per unit of area given by

$$V_0 = \frac{Eh}{2(1 - \nu^2)}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + 2\nu\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{1 - \nu}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2\right] \quad (2)$$

The Euler equations corresponding to (1) and (2) are the usual two-dimensional displacement equilibrium equations which in the present case are subject to the boundary conditions

$$u = v = 0 \quad \text{on } C \quad (3)$$

If X and Y are replaced by $[-E\alpha h(\partial T/\partial x)/1 - \nu]$ and $[-E\alpha h(\partial T/\partial y)/1 - \nu]$, respectively, then the problem is identical with that of plane stress due to temperature $T(x, y)$ and no body forces [2] When the thermal gradient in the thickness direction is not severe (in which case the Bernoulli-Euler hypothesis may be employed), the above variational problem is also analogous to the more general case for which $T = T(x, y, z)$ is an even function of z The quantities X and Y are then replaced by $(\partial N_T/\partial x)/1 - \nu$ and $-(\partial N_T/\partial y)/1 - \nu$, respectively, where $N_T = \int_h E\alpha T dz$

Solutions by the Rayleigh-Ritz method require that u and v be chosen in the form

$$u = \Sigma a_n f_n(x, y)$$
$$v = \Sigma b_n g_n(x, y) \quad (4)$$

where $f_n = g_n = 0$ on C In the case of a rectangular plate, for example, the displacements u and v may be expressed in the form of a complete double Fourier series:

$$u = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$
$$v = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \quad (5)$$

where a and b are the planform dimensions of the plate Sub-

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stitution of (5) into (1) and (2) yields the following infinite system of linear algebraic equations for the coefficients:

$$a_{rs}\left[2\left(\frac{r}{a}\right)^2 + (1-\nu)\left(\frac{s}{b}\right)^2\right]\pi^2 + \frac{4(1+\nu)rs}{ab}\sum_{\substack{m=1\,2\\ m\neq r}}^{\infty}\ \sum_{\substack{n=1\,2\\ n\neq s}}^{\infty}[1-(-1)^{m+r}]\times [1-(-1)^{n+s}]\frac{mnb_{mn}}{(m^2-r^2)(s^2-n^2)} = \frac{8(1-\nu^2)}{abEh}\int_0^a\int_0^b X\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dxdy \quad (6a)$$

$$b_{rs}\left[2\left(\frac{s}{b}\right)^2 + (1-\nu)\left(\frac{r}{a}\right)^2\right]\pi^2 + \frac{4(1+\nu)rs}{ab}\sum_{\substack{m=1\,2\\ m\neq r}}^{\infty}\ \sum_{\substack{n=1\,2\\ n\neq s}}^{\infty}[1-(-1)^{m+r}]\times [1-(-1)^{n+s}]\frac{mna_{mn}}{(n^2-s^2)(r^2-m^2)} = \frac{8(1-\nu^2)}{abEh}\int_0^a\int_0^b Y\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dxdy \quad (6b)$$

where $r = 1, 2,$ and $s = 1, 2,$ To obtain numerical solutions the series must be truncated

The foregoing problem was treated incorrectly in Ref 1

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Bending of Rhombic Plates

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Introduction

THE increased use of skew plates in aircraft structures and elsewhere appears to warrant the development of an adequate and easily applied method for their analysis in bending This analysis is complicated by the absence of orthogonality relationships and by the presence of possible singular behavior at obtuse corners, as described by Williams [1] In a recent paper, Morley[2] investigates the problem of a uniformly loaded, simply supported rhombic plate and points out the inadequacy of finite difference techniques for plates having large angles of skew He obtains solutions by introducing polar coordinates which will adequately display any singular behavior at the obtuse angle Certain boundary conditions are then satisfied in a least-square error sense The purpose of this note is to point out the computational advantages in applying the technique of "point matching"[3–5] to this problem and to compare the results obtained from this method with those of Morley [2] As a second example of "point matching," the uniformly loaded clamped rhombic plate is analyzed

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Simply Supported Plate

The rhombic plate geometry and polar coordinate system used by Morley is shown in Fig 1 The general solution of the uniformly loaded plate equation $\nabla^4 w = q_0/D$ which satisfies the boundary conditions of a plate simply supported along OA and OB is given in the form[2]

$$w = \frac{q_0}{D}\left\{\frac{r^4}{64}\left(1 - \frac{4\cos 2\theta}{3\cos 2\alpha} + \frac{\cos 4\theta}{3\cos 4\alpha}\right) + \sum_{n=1}^{\infty}(a_n + b_n r^2)r^{\lambda_n}\cos\lambda_n\theta\right\} \quad (1)$$

where

$$\lambda_n = (2n-1)\pi/2\alpha \quad (2)$$

As noted,[2] the first term of (1) must be suitably modified if $\alpha = \pi/8$, $\pi/4$, or $3\pi/8$ The arbitrary constants a_n and b_n are chosen ideally so as to satisfy symmetry boundary conditions along the diagonal AB These conditions demand

$$(\partial/\partial x)(\nabla^2 w)]_{AB} = 0 \qquad \partial w/\partial x]_{AB} = 0 \quad (3)$$

which give

$$\left\{a\cos\alpha\left(1 - \frac{1}{\cos 2\alpha}\right) + 8\sum_{n=1}^{\infty}\times \lambda_n(\lambda_n+1)b_n r^{\lambda_n - 1}\cos(\lambda_n - 1)\theta\right\}_{AB} = 0 \quad (4)$$

$$\left\{\frac{r^3}{16}\left[\left(1 - \frac{1}{\cos 2\alpha}\right)\cos\theta + \frac{1}{3}\left(\frac{1}{\cos 4\alpha} - \frac{1}{\cos 2\alpha}\right)\cos 3\theta\right] + \sum_{n=1}^{\infty}\lambda_n a_n r^{\lambda_n - 1}\cos(\lambda_n - 1)\theta + \sum_{n=1}^{\infty} b_n r^{\lambda_n + 1}\times [(\lambda_n + 1)\cos(\lambda_n - 1)\theta + \cos(\lambda_n + 1)\theta]\right\}_{AB} = 0 \quad (5)$$

In the absence of orthogonality relations, (4) and (5) can be satisfied only approximately by a finite number N of terms In Ref 2, the a_n and b_n are chosen so as to minimize the integral square error over AB This leads to numerous integrals which are rather tedious to evaluate, and then finally to the solution of two sets of N by N simultaneous equations The method of "point matching" requires satisfying conditions (4) and (5) exactly at N discrete points along AB This leads directly to two sets of N by N simultaneous equations for determining the a_n and b_n The singular behavior at the obtuse angle is identical for both methods of solution

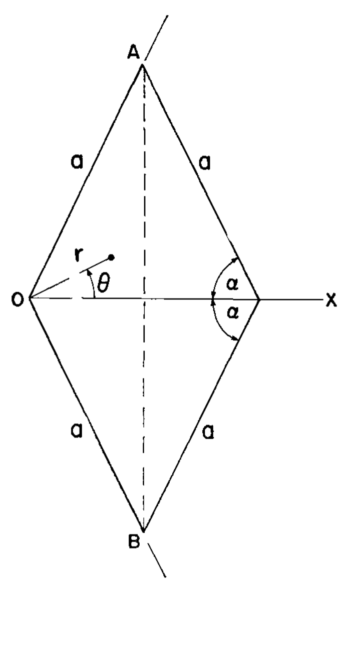

Fig 1 Rhombic plate